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A lattice implementation of the η-invariant and effective action for chiral fermions on the lattice

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Abstract

We consider a lattice implementation of the η-invariant, using the complex phase of the determinant of the simplified domain-wall fermion, which couples to an interpolating five-dimensional gauge field. We clarify the relation to the effective action for chiral Ginsparg-Wilson fermions. The integrability, which holds true for anomaly-free theories in the classical continuum limit, is not assured on the lattice with a finite spacing. A lattice expression for the five-dimensional Chern-Simons term is obtained.

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1 Introduction

It has become clear recently that the gauge interaction of the Weyl fermions can be described in the framework of lattice gauge theory. The clue to this development is the construction of gauge covariant and local Dirac operators [1, 2, 3] which solve the Ginsparg-Wilson relation [4]. The Ginsparg-Wilson relation implies the exact chiral symmetry for the Dirac fermion [5] and suggests an asymmetric and gauge-field-dependent chiral projection to the Weyl degrees of freedom [6, 7]. The functional measure for the Weyl fermion field is defined based on the chiral projection. It leads to a mathematically reasonable definition of the chiral determinant, which generally has the structure as an overlap of two vacua [8]. It has been shown by Lüscher in [9, 10] that for anomaly-free abelian chiral gauge theories, the functional measure for the Weyl fermion fields can be constructed so that the gauge invariance is maintained exactly on the lattice.

On the other hand, in the continuum theory, Alvarez-Gaumé et al. [11] and Ball and Osborn [12] have shown that the imaginary part of the effective action for chiral fermions can be given by the η-invariant [13]. It is defined as the spectrum asymmetry of the five-dimensional massless Dirac operator coupled to an interpolating five-dimensional gauge field.

In this paper, we will show that a lattice implementation of the η-invariant is possible so that the lattice η-invariant gives the imaginary part of the effective action for the chiral Ginsparg-Wilson fermion defined by Neuberger's Dirac operator. We define the η-invariant on the lattice using the complex phase of the determinant of the (simplified) domain-wall fermion [14, 15], which couples to an interpolating five-dimensional gauge field. Our formulation then can be regarded as a lattice realization of the argument given by Kaplan and Shmaltz in the continuum theory [16], using the simplified formulation of the domain-wall fermion by Shamir [15].

Our lattice implementation of the η-invariant can be shown to have a direct relation to the imaginary part of the effective action for the chiral Ginsparg-Wilson fermions which is defined by Neuberger's Dirac operator [8, 10, 17]. This implementation is applicable to non-abelian chiral gauge theories. But the integrability, which holds true for anomaly-free theories in the classical continuum limit, is not assured on the lattice with a finite spacing. This issue of the integrability for anomaly free chiral gauge theories is discussed. A lattice expression for the five-dimensional Chern-Simons term is obtained. [1]

[1]When completing this work and preparing this article, we noticed that a paper by Lüscher [18] appeared. In [18], a formula of the effective action for the chiral Ginsparg-

This paper is organized as follows. The section 2 is devoted to reviews of effective action for chiral fermions in the continuum theory and on the lattice: In section 2.1, we first review the relation bewteen the effective action for chiral fermions and the η-invariant in the continuum theory. In section 2.2, the effective action for the chiral Ginsparg-Wilson fermion defined through Neuberger's lattice Dirac operator is reviewed. In section 2.3, we discuss the relation between Neuberger's Dirac operator and the domain-wall fermion for vector-like theories. In section 3, we describe our implementation of the η-invariant on the lattice, using chiral domain-wall fermion. In section 4, we examine the variation of the lattice η-invariant with respect to the gauge field. In section 5, we clarify the relation of the lattice η-invariant to the effective action for chiral Ginsparg-Wilson fermions defined through Neuberger's lattice Dirac operator. The integrability for anomaly free chiral gauge theories, which is not assured a priori on the lattice, is discussed. In section 6, we summarize our result and give some discussions.

In the following discussions, various fermion theories are considered in the continuum limit, and on four- and five- dimensional lattices. The four-dimensional space-time coordinates are denoted by $x_\mu (\mu = 1, 2, 3, 4)$ and the fifth-dimensional coordinate is denoted by t. We denote the lattice spacing of the four direction $\mu = 1, 2, 3, 4$ with a and the lattice spacing of the fifth direction with a_5. Then the lattice indices are given as follows:

$$\begin{aligned} x_\mu &= n_\mu\, a, \quad n_\mu \in \mathbb{Z} \quad (\mu = 1, 2, 3, 4), & (1.1)\\ t &= n_5\, a_5, \quad n_5 \in \mathbb{Z}. & (1.2) \end{aligned}$$

The gauge-covariant difference operators are defined with link variables as

$$\begin{aligned} \nabla_\mu \phi(x,t) &= \frac{1}{a}\left(U_\mu(x,t)\phi(x+\hat{\mu}a,t) - \phi(x,t) \right), \quad (\mu = 1, 2, 3, 4) & (1.3)\\ \nabla_5 \phi(x,t) &= \frac{1}{a_5}\left(U_5(x,t)\phi(x,t+a_5) - \phi(x,t) \right), & (1.4) \end{aligned}$$

where the unit vector in the direction μ is denoted by $\hat{\mu}$.

2 Effective action for chiral fermions in the continuum theory and on the lattice

Wilson fermion which couples to non-abelian gauge fields is derived and its relation to the η-invariant is suggested.

2.1 The η-invariant in the continuum theory

In the continuum theory, the η-invariant [13] is defined as the spectrum asymmetry of the hermitian five-dimensional Dirac operator. It can be defined through the complex phase of the determinant of a five-dimensional massless Dirac fermion in the Pauli-Villars regularization [11, 19].

$$\pi\eta = -\mathrm{Im}\log\det\left[(H - iM)\,/\,(H + iM)\right], \qquad H \equiv i\sum_{M=1}^{5}\gamma_M D_M. \tag{2.1}$$

In this formula, it is assumed that the five-dimensional Dirac operator couples to gauge fields $A_M(x,t)$ with the following property: for $t = -\infty$ to $-\Delta$, $A_M(x,t) = A^0_M(x)$; for $t = -\Delta$ to $+\Delta$, $A_M(x,t)$ smoothly interpoltes between $A^0_M(x)$ and $A^1_M(x)$ and from $t = +\Delta$ to $+\infty$, $A_M(x,t) = A^1_M(x)$. Both $A^0_M(x)$ and $A^1_M(x)$ are assumed to be perturbative configurations and $A_M(x,t)$ can be chosen so that the five-dimensional Dirac operator does not have zero modes.

In [11], the variation of the η-invariant with respect to gauge field has been examined, by introducing one more parameter u which parametrizes the gauge configuration for $t \geq +\Delta$. The result can be written as the sum of the four-dimensional surface contribution and the five-dimensional bulk contribution:

$$\begin{aligned}\pi\frac{d}{du}\eta \;&=\; \mathrm{Im}\,\mathrm{Tr}_x\, P_L\,\frac{d}{du}\not{D}\frac{1}{\not{D}} \\ &-\; \lim_{T\to\infty}\int d^4x\int_{-T}^{T} dt\,\frac{1}{32\pi^2}\,\epsilon_{\mu MNKL}\mathrm{Tr}\left\{\frac{d}{du}A_\mu\,F_{MN}F_{KL}\right\}(x,t;u).\end{aligned} \tag{2.2}$$

The first term has the form of the gauge current induced by chiral fermions which is regularized gauge-covariantly. The second five-dimensional term can be written as the variation of the Chern-Simons term [11], up to the local current of Bardeen and Zumino [20]:

$$-\frac{d}{du}\left\{2\pi Q_5\left[A_\mu(x,t;u)\right]\right\} + \int d^4x\,\mathrm{tr}\left\{\frac{d}{du}A_\mu(x;u)\,X_\mu(x;u)\right\}, \tag{2.3}$$

where

$$\begin{aligned}&2\pi Q_5\left[A_\mu\right] \\ &\quad= \lim_{T\to\infty}\int\int_{-T}^{T} d^4x\,dt\,\int_0^1 d\sigma\,\frac{1}{32\pi^2}\epsilon_{\mu MNKL}\mathrm{Tr}\left\{A_\mu F^\sigma_{MN}F^\sigma_{KL}\right\},\end{aligned} \tag{2.4}$$

$$F^\sigma_{MN} = \sigma\left(\partial_M A_N - \partial_N A_M\right) + \sigma^2 i\left[A_M, A_N\right], \tag{2.5}$$

and

$$X_\mu = \frac{1}{48\pi^2}\,\epsilon_{\mu\nu\lambda\rho}\left(A_\nu F_{\lambda\rho} + F_{\nu\lambda}A_\rho - A_\nu A_\lambda A_\rho\right). \tag{2.6}$$

The current of Bardeen and Zumino, denoted by X_μ here, plays the role to translate the covariant gauge current, which is induced from the surface term, to the consistent current [11].

Then integrating the expression with respect to u,

$$\begin{aligned} \pi\frac{d}{du}\eta &= \mathrm{Im}\int d^4x\,\mathrm{Tr}\; i\frac{d}{du}A_\mu \left\{\mathrm{tr}\left(\frac{1}{2}\gamma_\mu\gamma_5\,\frac{1}{D\!\!\!\!/}\right) + X_\mu\right\}\left[A_\mu(x;u)\right] \\ &- \frac{d}{du}\left\{2\pi Q_5\left[A_\mu(x,t)\right]\right\}, \end{aligned} \tag{2.7}$$

we obtain

$$\mathrm{Im}\,\Gamma_{\text{eff}}\left[A^1_\mu\right] - \mathrm{Im}\,\Gamma_{\text{eff}}\left[A^0_\mu\right] = \pi\eta + 2\pi Q_5\left[A_\mu(x,t)\right]. \tag{2.8}$$

We note here the role of the Chern-Simons term. 1) First of all, the Chern-Simons term compensates the dependence of the η-invariant on the path of the interpolation and make it integrable so that it can give the effective action of chiral fermions which depends only the values of gauge fields at the boundaries. 2) The Chern-Simons term reproduces the non-abelian gauge anomaly of the effective action, while the η-invariant is gauge invariant. If $A^1_\mu(x)$ is obtained from $A^0_\mu(x)$ by a certain gauge tranformation,

$$A^1_\mu(x) = g(x)A^0_\mu(x)g(x)^{-1} - ig(x)\partial_\mu g(x)^{-1}, \tag{2.9}$$

we may consider an interpolation of the gauge transformation function, $g(x,t)$, such that $g(x, t = -\infty) = 1$ and $g(x, t = \infty) = g(x)$ and the region of the interpolation is within $t \in [-\Delta, \Delta]$. Then we obtain

$$\begin{aligned} &\mathrm{Im}\,\Gamma_{\text{eff}}\left[g(x)A^0_\mu(x)g(x)^{-1} - ig(x)\partial_\mu g(x)^{-1}\right] - \mathrm{Im}\,\Gamma_{\text{eff}}\left[A^0_\mu(x)\right] \\ &\qquad = 2\pi Q_5\left[g(x,t)A^0_\mu(x)g(x,t)^{-1} - ig(x,t)\partial_\mu g(x,t)^{-1}; -ig(x,t)\partial_5 g(x,t)^{-1}\right]. \end{aligned} \tag{2.10}$$

The r.h.s. is nothing but the Wess-Zumino action. 3) When the non-abelian gauge anomaly is canceled by the condition

$$\sum_R \mathrm{Tr}_R\left(T^a\left\{T^b, T^c\right\}\right) = 0, \tag{2.11}$$

the Chern-Simons term vanishes. The η-invariant becomes integrable and identical to the imaginary part of the gauge invariant effective action for chiral fermions.

$$\mathrm{Im}\,\Gamma_{\mathrm{eff}}\left[A_\mu\right] - \mathrm{Im}\,\Gamma_{\mathrm{eff}}\left[A_\mu^0\right] = \pi\eta. \tag{2.12}$$

Thus the imaginary part of the effective action of chiral fermions can be expressed through the η-invariant.

Since $X_\mu(x;u)$ is orthogonal to $A_\mu(x;u)$, it does not contribute in the integration of u if we adopt the linear interpolation as $A_\mu(x;u) = uA_\mu(x)$. In this case, the imaginary part of the effective action is entirely given by the integration of the covariant gauge current induced from the surface term [21].

$$\mathrm{Im}\,\Gamma_{\mathrm{eff}}\left[A_\mu\right] - \mathrm{Im}\,\Gamma_{\mathrm{eff}}\left[A_\mu^0\right] = \int_0^1 du\, \mathrm{Im}\,\mathrm{Tr}_x\, P_L \frac{d}{du} \not{D} \frac{1}{\not{D}}. \tag{2.13}$$

The integration of the bulk term gives directly the Chern-Simons term.

$$\begin{aligned} &2\pi Q_5\left[A_\mu(x,t;u)\right] \\ &= \int_0^1 du \lim_{T\to\infty} \int\int_{-T}^{T} d^4x dt\, \frac{1}{32\pi^2}\, \epsilon_{\mu MNKL} \mathrm{Tr}\left\{ \frac{d}{du} A_\mu\, F_{MN} F_{KL} \right\}(x,t;u). \end{aligned} \tag{2.14}$$

2.2 Effective action for chiral Ginsparg-Wilson fermion defined by Neuberger's lattice Dirac operator

2.2.1 Neuberger's lattice Dirac operator

Neuberger's lattice Dirac operator [1], which is derived from the overlap formalism [8] and satisfies the Ginsparg-Wilson relation [4], is given by the following formula:

$$D = \frac{1}{2a}\left(1 + X \frac{1}{\sqrt{X^\dagger X}}\right) = \frac{1}{2a}\left(1 + \gamma_5 \frac{H}{\sqrt{H^2}}\right), \tag{2.15}$$

where X is the Wilson-Dirac operator with a negative mass and H is the hermitian operator,

$$X = D_{\mathrm{w}} - \frac{m_0}{a}, \qquad H = \gamma_5\left(D_w - \frac{m_0}{a}\right), \qquad (0 < m_0 < 2), \tag{2.16}$$

$$D_{\mathrm{w}} = \sum_{\mu=1}^{4} \left\{ \gamma_\mu \frac{1}{2} \left(\nabla_\mu - \nabla_\mu^\dagger \right) + \frac{a}{2} \nabla_\mu \nabla_\mu^\dagger \right\}. \tag{2.17}$$

This Dirac operator satisfies the Ginsparg-Wilson relation [4].

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D. \tag{2.18}$$

Locality properties of Neuberger's lattice Dirac operator has been examined by Hernándes, Jansen and Lüscher [3]. For a certain class of gauge fields with small lattice field strength, exponential bounds have been proved rigorously on the kernels of the Dirac operator and its differentiations with respect to the gauge field. Namely, if the field strength of the gauge field bounded uniformly as follows,

$$\| 1 - U_{\mu\nu}(x) \| < \epsilon, \qquad \epsilon < \frac{1}{30} \left\{ 1 - |1 - m_0|^2 \right\}, \tag{2.19}$$

then the Wilson-Dirac operator square is bounded below by a positive constant as

$$\left\| a^2 \left(D_{\mathrm{w}} - \frac{m_0}{a} \right)^\dagger \left(D_{\mathrm{w}} - \frac{m_0}{a} \right) \right\| \geq \left\{ (1 - 30\epsilon)^{\frac{1}{2}} - |1 - m_0| \right\}^2. \tag{2.20}$$

Given the positive lower and upper bounds for $a^2 X^\dagger X$, it follows that the kernel of Neuberger's Dirac operator is exponentially bounded as

$$a^4 \parallel D(x,y) \parallel = C \exp \left\{ -\frac{\theta}{2a} |x - y| \right\}, \tag{2.21}$$

where θ is a certain constant which is determined from the lower and upper bounds for $a^2 X^\dagger X$.

2.2.2 Effective action for chiral Ginsparg-Wilson fermion

The Ginsparg-Wilson relation Eq. (2.18) implies the exact symmetry of the fermion action. For the Dirac fermion described by the lattice Dirac operator which satisfies the Ginsparg-Wilson relation

$$S_D = a^4 \sum_x \bar{\psi}(x) D \psi(x), \tag{2.22}$$

chiral transformation can be defined as follows:

$$\delta\psi(x) = \gamma_5 \left(1 - 2aD \right) \psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_5. \tag{2.23}$$

Then it is straightforward to see that the action is invariant under this transformation.

From this property, Weyl fermion can be introduced as the eigenstate of the generators of the chiral transformation

$$\hat{\gamma}_5 = \gamma_5 \left(1 - 2aD\right), \qquad \gamma_5. \tag{2.24}$$

Namely, the right-handed Weyl fermion is defined through the constraint given as follows [6, 10]:

$$\hat{P}_R \psi_R(x) = \psi_R(x), \quad \bar{\psi}_R(x) P_L = \bar{\psi}_R(x), \tag{2.25}$$

where $\hat{P}_R$ is the chiral projector for the fermion field $\psi(x)$ defined as

$$P_R = \left(\frac{1+\hat{\gamma}_5}{2}\right), \qquad P_L = \left(\frac{1-\gamma_5}{2}\right). \tag{2.26}$$

The action of the Weyl fermion is given by

$$S_W = a^4 \sum_x \bar{\psi}_R(x) D \psi_R(x). \tag{2.27}$$

The functional integral measure for the Weyl fermion can be defined by introducing the chiral basis,

$$\hat{P}_R v_j(x) = v_j(x), \qquad (v_k, v_j) = \delta_{kj}, \tag{2.28}$$

where $(v_k, v_j) = a^4 \sum_x v_k(x)^\dagger v_j(x)$. Using this chiral basis, the Weyl fermion can be expanded with the coefficients c_j which generates a Grassmann algebra. Then the functional measure for the right-handed field is given by

$$D\left[\psi\right] = \prod_j dc_j \qquad \psi_R(x) = \sum_j v_j(x) c_j. \tag{2.29}$$

The functional measure for the anti-fermion field is defined similarly with the basis

$$\bar{v}_k(x) P_L = \bar{v}_k(x), \qquad (\bar{v}_k, \bar{v}_j) = \delta_{kj}, \tag{2.30}$$

as follows:

$$D\left[\bar{\psi}\right] = \prod_k d\bar{c}_k \qquad \bar{\psi}_R(x) = \sum_k \bar{c}_k \bar{v}_k(x). \tag{2.31}$$

Then the partition function of the Weyl fermion is given as

$$Z_F = \int D[\psi]\, D[\bar{\psi}]\, e^{-S_W[\psi_R, \bar{\psi}_R]} = \det M_{kj}, \tag{2.32}$$

where

$$M_{kj} = \bar{v}_k D v_j. \tag{2.33}$$

In the case using Neuberger's Dirac operator, the right-handed projector is noting but the projector to negative energy states of the Wilson-Dirac hamiltonian H,

$$\hat{P}_R = \frac{1 - \frac{H}{\sqrt{H^2}}}{2}. \tag{2.34}$$

Then the basis $v_j(x)$ can be chosen as the normalized complete set of the eigenvectors of negative energy states. The basis $\bar{v}_k(x)$ may be regarded as the complete set of the negative-energy eigenvectors of the hamiltonian γ_5. Then the chiral determinant may be written in the form of the overlap of two vacua [8]:

$$\det M_{kj} = \det\left(\bar{v}_k v_j\right). \tag{2.35}$$

It is useful for later discussions to consider the variation of the effective action with respect to the gauge fields [22, 10, 18]. Following [10, 18], we write the variation of the link variables as

$$\delta_\zeta U_\mu(x) = a\zeta_\mu(x) U_\mu(x), \qquad \zeta_\mu(x) = iT^a \zeta_\mu^a(x). \tag{2.36}$$

Then the variation of the effective action is evaluated as

$$\delta_\zeta \ln\det M_{kj} = \mathrm{Tr}\delta_\zeta D \hat{P}_R D^{-1} P_L + \sum_k \left(v_k, \delta_\zeta v_k\right). \tag{2.37}$$

The second term of the r.h.s. is discussed in [22] in analogy of the Berry connection. It is refered as the measure term in [10, 18]. This term may be expressed with a current as follows:

$$\sum_k \left(v_k, \delta_\zeta v_k\right) = -ia^4 \sum_x \zeta_\mu^a(x) j_\mu^a(x). \tag{2.38}$$

2.2.3 Gauge invariant choice of the measure term in abelian chiral gauge theories

It has been shown by Lüscher in [9, 10] that for anomaly-free abelian chiral gauge theories, the functional measure for the Weyl fermion fields can be constructed so that the gauge invariance of the effective action is maintained exactly on the lattice. If we consider a gauge transformation of the effective action in an abelian chiral gauge theory, we obtain

$$\delta_\omega \ln \det M_{kj} = i \sum_x \omega(x) \left\{ \mathrm{tr} T \gamma_5 \left(1 - aD\right)(x,x) - a^4 \partial_\mu^* j_\mu(x) \right\}. \tag{2.39}$$

In this respect, the result obtained by Lüscher [9], which is crucial for the gauge invariance, is that the anomaly associated with the chiral gauge current can be expressed in the following form:

$$\mathrm{tr}\gamma_5 \left(1 - aD\right)(x,x) = a^4 \left\{ \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_\mu^* \bar{k}_\mu(x) \right\}, \tag{2.40}$$

where $\bar{k}_\mu(x)$ is a gauge-invariant local current. This is the consequence of the index theorem on the lattice [23]. Then it has been shown that the basis $\{v_j(x)\}$ can be chosen so that the current associated with the measure term is gauge-invariant and local and it satisfies the anomalous conservation law,

$$\partial_\mu^* j_\mu(x) = \frac{1}{a^4} \mathrm{tr}\gamma_5 \left(1 - aD\right)(x,x) = \partial_\mu^* \bar{k}_\mu(x), \tag{2.41}$$

when the anomaly cancellation condition $\sum_i e_i^3 = 0$ is satisfied. The explicit form of the ansatz for the measure term is given by

$$\begin{aligned} - i a^4 \sum_k \zeta_\mu(x) j_\mu(x) \quad &= \quad -i \int_0^1 dt \mathrm{Tr} \left\{ \hat{P}_R \left[\partial_t \hat{P}_R, \delta_\zeta \hat{P}_R \right] \right\} \\ &\quad - \int_0^1 dt \sum_x \left\{ \zeta_\mu(x) \bar{k}_\mu(x) + A_\mu(x) \delta_\zeta \bar{k}_\mu(x) \right\}, \end{aligned} \tag{2.42}$$

where $U_\mu(x,t) = \exp(it A_\mu(x))$.

2.3 Domain-wall fermion for vector-like theories

Neuberger's lattice Dirac operator has a close relation to the domain-wall fermion [24, 25, 26]. In a simplified formulation, the domain-wall fermion is

defined by the five-dimensional Wilson fermion with the Dirichlet boundary condition in the fifth dimension. With the Dirichlet boundary condition in the fifth direction,

$$\psi_R(x,t)|_{t=-T} = 0, \quad \psi_L(x,t)|_{t=T+a_5} = 0, \qquad (T = Na_5), \tag{2.43}$$

the action of the domain-wall fermion is defined by

$$S_{\rm DW} = a_5 \sum_{t=-T+a_5}^{T} a^4 \sum_x \bar\psi(x,t) \left(D_{5\rm w} - \frac{m_0}{a} \right) \psi(x,t), \quad (0 < m_0 < 2), \tag{2.44}$$

where the gauge field is asuumed to be four-dimensional

$$U_\mu(x,t) = U_\mu(x), \quad U_5(x,t) = 1, \tag{2.45}$$

and the five-dimensional Wilson-Dirac operator $D_{5\rm w}$ is defined as

$$D_{5\rm w} \;=\; \sum_{\mu=1}^{4} \left\{ \gamma_\mu \frac{1}{2} \left(\nabla_\mu - \nabla_\mu^\dagger \right) + \frac{a}{2} \nabla_\mu \nabla_\mu^\dagger \right\} + \gamma_5 \frac{1}{2} \left(\nabla_5 - \nabla_5^\dagger \right) + \frac{a_5}{2} \nabla_5 \nabla_5^\dagger. \tag{2.46}$$

Due to its structure of the chiral hopping and the boundary condition in the fifth dimension, a single light Dirac fermion can emerge in the spectrum. This light fermion can be probed suitably by the field variables at the boundary of the fifth dimension, which are referred as $q(x)$ and $\bar q(x)$ by Furman and Shamir.

$$q(x) = \psi_L(x, -T + a_5) + \psi_R(x, T), \qquad \bar q(x) = \bar\psi_L(x, -T+1) + \bar\psi_R(x, T). \tag{2.47}$$

In fact, the propagator of the light fermion field can be expressed in terms of the effective Dirac operator [24, 26]:

$$\langle q(x) \bar q(y) \rangle = \frac{1}{a^4} \left(\frac{1}{a} D_{\rm eff}^{(T)\,-1} - \delta(x,y) \right), \tag{2.48}$$

where

$$D_{\rm eff}^{(T)} = \frac{1}{2a} \left(1 + \gamma_5 \tanh T \widetilde H \right), \qquad (T = Na_5). \tag{2.49}$$

$\widetilde{H}$ is defined through the transfer matrix of the five-dimensional Wilson fermion

$$e^{-a_5\tilde{H}} = \begin{pmatrix} \frac{1}{B} & -\frac{1}{B}C \\ -C^\dagger\frac{1}{B} & B + C^\dagger\frac{1}{B}C \end{pmatrix}, \tag{2.50}$$

where [2]

$$\begin{aligned} C &= a_5\,\sigma_\mu\,\frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right), && (2.51)\\ B &= 1 + a_5\left(-\frac{a}{2}\nabla_\mu\nabla_\mu^* - \frac{m_0}{a}\right). && (2.52) \end{aligned}$$

The limit $T \to \infty$ is defined well as long as $\widetilde{H}^2 > 0$. The effective Dirac operator Eq. (2.49) then reduces to Neuberger's lattice Dirac operator using $\widetilde{H}$,

$$D_{\text{eff}} = \frac{1}{2a}\left(1 + \gamma_5\frac{\widetilde{H}}{\sqrt{\widetilde{H}^2}}\right), \tag{2.53}$$

and turns out to satisfy the Ginsparg-Wilson relation.

It is useful to note that the effective Dirac operator admits the following representation [26, 27]:

$$\begin{aligned} aD_{\text{eff}}^{(T)} = {} & 1 - P_R\left\{a_5\left(\overline{D}_{5\text{w}} - \frac{m_0}{a}\right)\right\}^{-1}_{T,T} P_L \\ & -P_L\left\{a_5\left(\overline{D}_{5\text{w}} - \frac{m_0}{a}\right)\right\}^{-1}_{-T+a_5,-T+a_5} P_R \\ & -P_R\left\{a_5\left(\overline{D}_{5\text{w}} - \frac{m_0}{a}\right)\right\}^{-1}_{T,-T+a_5} P_R \\ & -P_L\left\{a_5\left(\overline{D}_{5\text{w}} - \frac{m_0}{a}\right)\right\}^{-1}_{-T+a_5,T} P_L, \end{aligned} \tag{2.54}$$

where $\overline{D}_{5\text{w}}$ is the five-dimensional Wilson-Dirac operator *with the anti-periodic boundary condition* in the fifth-dimension. Its inverse may be expressed as

$$\left\{a_5\left(\overline{D}_{5\text{w}} - \frac{m_0}{a}\right)\right\}^{-1}_{st} = \frac{1}{2N}\sum_p \frac{e^{ip(s-t)}}{i\gamma_5\sin pa_5 + 1 - \cos pa_5 + a_5\left(D_\text{w} - \frac{m_0}{a}\right)}. \tag{2.55}$$

[2] In this expression, the positivity of B is required for the transfer matrix to be defined consistently. It is assured when $0 < \frac{a_5}{a} m_0 < 1$.

The summation is taken over the discrete momenta $p = \frac{\pi}{Na_5}(k - \frac{1}{2})$ $(k = 1, 2, \cdots, 2N)$ and in the limit $N \to \infty$ it reduces to the continuous integral.

From this representation, it is rather clear that the effective Dirac operator can be defined consistently if the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition is not singular and invertible for all T. In this respect, we should note that the lower bound on the square of the five-dimensional Wilson-Dirac operator is related closely to that on the square of the four-dimensional Wilson-Dirac operator [3, 28], because the gauge field is four-dimensional. In fact, the same lower bound can be set for the class of gauge fields with small lattice field strength which satisfies the bound Eq. (2.19).

$$\left\| a^2 \left(\overline{D}_{5\mathrm{w}} - \frac{m_0}{a} \right)^\dagger \left(\overline{D}_{5\mathrm{w}} - \frac{m_0}{a} \right) \right\| \geq \left\{ (1 - 30\epsilon)^{\frac{1}{2}} - |1 - m_0| \right\}^2 . \qquad (2.56)$$

3 A lattice implementation of the η-invariant

3.1 The η-invariant, the five-dimensional massless fermion on the lattice and the chiral domain-wall fermion

In the continuum theory, the η-invariant can be defined through the complex phase of the determinant of a five-dimensional massless Dirac fermion in the Pauli-Villars regularization [11, 19]. In order to implement the η-invariant on the lattice, we may consider a five-dimensional massless Dirac fermion which is formulated on the lattice using the five-dimensional overlap Dirac operator [29]. As shown in [29], in the overlap formalism [8], a five-dimensional massless Dirac fermion can be described gauge invariantly by the five-dimensional overlap Dirac operator:

$$S_{5\mathrm{dim}} = a_5 a^4 \sum_{x,t} \bar{\psi}(x,t) \left(1 + X_5 \frac{1}{\sqrt{X_5^\dagger X_5}} \right) \psi(x,t), \qquad (3.1)$$

where X_5 denotes the five-dimensional Wilson-Dirac operator $D_{5\mathrm{w}}$ with a negative mass:

$$X_5 = D_{5\mathrm{w}} - \frac{m_0}{a} \qquad (0 < m_0 < 2). \qquad (3.2)$$

Then we can consider a lattice implementation of the η-invariant using the complex phase of the determinant of the five-dimensional massless Dirac

fermion which couples to a certain interpolating five-dimensional lattice gauge field:

$$\frac{\pi}{2}\,\overline{\eta} \;\simeq\; \mathrm{Im}\ln\det\left(1+X_5\frac{1}{\sqrt{X_5^\dagger X_5}}\right). \tag{3.3}$$

Since it holds that [29] [3]

$$\mathrm{Im}\ \ln\ \det\left(1+X_5\frac{1}{\sqrt{X_5^\dagger X_5}}\right)=\frac{1}{2}\mathrm{Im}\ \ln\ \det X_5, \tag{3.5}$$

the above implementation can also be written as

$$\pi\,\overline{\eta}\simeq \mathrm{Im}\ \ln\ \det\left(D_{5\mathrm{w}}-\frac{m_0}{a}\right). \tag{3.6}$$

These consideration suggests that the η-invariant can be implemented on the lattice through the complex phase of the determinant of the simplified domain-wall fermion which couples to the interpolating five-dimensional gauge field. Namely, we first impose the Dirichlet boundary condition in the fifth dimension and then let the extent of the fifth dimension go to infinity.

$$\pi\,\overline{\eta}\simeq \lim_{T\to\infty}\mathrm{Im}\ \ln\ \det\left(D_{5\mathrm{w(T)}}-\frac{m_0}{a}\right). \tag{3.7}$$

$D_{5\mathrm{w(T)}}$ stands for the Wilson-Dirac operator which is subject to the Dirichlet boundary condition at $t=-T+a_5$ and $t=T$.[4] Because of the interpolation in the fifth direction, the chiral modes at the boundary $t=-T+a_5$ and $t=T$ couple to different four-dimensional gauge fields. This difference is expected to reprodue the difference of the complex phase of the effective

[3] If we use an abbreviation as $V=X_5\frac{1}{\sqrt{X_5^\dagger X_5}}$, we have $V^\dagger V=1$. Then, the complex phase of the partition function of the five-dimensional massless Dirac fermion can be evaluated as

$$\mathrm{Im}\ln\det{}_{(T)}\left(1+V\right)=\frac{1}{2i}\ln\det{}_{(T)}\frac{(1+V)}{(1+V^\dagger)}=\frac{1}{2i}\ln\det{}_{(T)}\frac{1}{V^\dagger}=\frac{1}{2}\mathrm{Im}\ln\det{}_{(T)}V. \tag{3.4}$$

Since $\sqrt{X_5^\dagger X_5}$ is hermitian, it does not contribute to the complex phase of the partition function and can be neglected. Thus we obtain Eq. (3.5).

[4] With this boundary condition, the difference operator in the fifth direction

$$\partial_5=\frac{1}{a_5}\left(\delta_{x+\hat{5},y}-\delta_{x,y}\right) \tag{3.8}$$

action for the chiral Ginsparg-Wilson fermion, since the chiral modes at the boundaries are described by the effective Dirac operator satisfying the Ginsparg-Wilson relation. Our proposal then can be regarded as a lattice realization of the argument given by Kaplan and Shmaltz in [16], using the simplified domain-wall fermion of Shamir [15].

3.2 Smooth interpolation of four-dimensional gauge fields

In order to realize the smooth interpolation of the four-dimensional lattice gauge fields, we need to choose carefully the interpolating five-dimensional gauge field. For this purpose, we first consider the five-dimensional lattice theory Eq. (3.7) and then take the continuum limit in the fifth dimension, $a_5 \to 0$.

To prepare the interpolating gauge fields on the five-dimensional lattice, let us consider two four-dimensional gauge fields.

$$U_\mu(x) = e^{iA_\mu(x)}, \qquad U^0_\mu(x) = e^{iA^0_\mu(x)}. \tag{3.10}$$

We assume that both gauge fields are smooth enough to satisfies the bound Eq. (2.19) to make Neuberger's Dirac operator defined well and local [3]. We also assume that both gauge fields belong to the same topological sector in which the topological charge defined through Neuberger's Dirac operator

$$Q = -a\mathrm{Tr}\gamma_5 D = -\frac{1}{2}\mathrm{Tr}\frac{H}{\sqrt{H^2}} \tag{3.11}$$

are equal.

Then we consider a five-dimensional gauge field interpolating two four-dimensional gauge fields, $U^0_\mu(x)$ and $U_\mu(x)$, along the fifth coordinate t which is for the first time regarded as a continuous coordinate.

$$U_\mu(x,t) \qquad t \in R. \tag{3.12}$$

---

may be expressed by $2N \times 2N$ matrix ($T = Na_5$) as follows:

$$\partial_{5(T)} = \frac{1}{a_5}\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad (N=3). \tag{3.9}$$

The subscript (T) denotes the fact that the difference operator is implemented by a finite matrix, taking account of the Dirichlet boundary condition in the fifth direction.

We assume that it is possible to choose such an interpolation without breaking the bound Eq. (2.19) and changing the topological property of the gauge fields:

$$\| 1 - U_{\mu\nu}(x,t) \| < \epsilon, \qquad \epsilon < \frac{1}{30} \left\{ 1 - |1 - m_0|^2 \right\}, \quad t \in R. \tag{3.13}$$

We also assume that the interpolating region has a finite interval, $t \in [-\Delta, \Delta]$: When $t < -\Delta$, it coincides with the four-dimensional gauge field $U^0_\mu(x)$,

$$U_\mu(x,t) \underset{t<-\Delta}{\longrightarrow} U^0_\mu(x). \tag{3.14}$$

When $t > \Delta$, it coincides with the other four-dimensional gauge field $U_\mu(x)$:

$$U_\mu(x,t) \underset{t>+\Delta}{\longrightarrow} U_\mu(x). \tag{3.15}$$

This defines one parameter family of the interpolating five-dimensional gauge fields. See Figure 1.

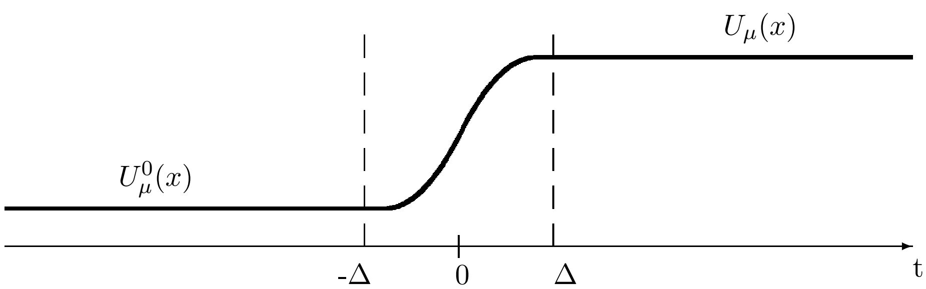

Figure 1: Interpolating five-dimensional gauge field

We then map the continuum interpolations to a discrete fifth dimensional lattice space so that $\Delta < T$. (Figure 2)

$$U_\mu(x,t) \qquad t = n_5 a_5, \quad n_5 \in Z. \tag{3.16}$$

This interpolating five-dimensional lattice gauge field is to couple to the domain-wall fermion of Eq. (3.7). In order to recover the smooth interpolation of the two four-dimensional lattice gauge fields, we need to take both the infinite extent limit $T \to \infty$ and the continuum limit $a_5 \to 0$ of the fifth dimension, keeping $\Delta \ll T$.

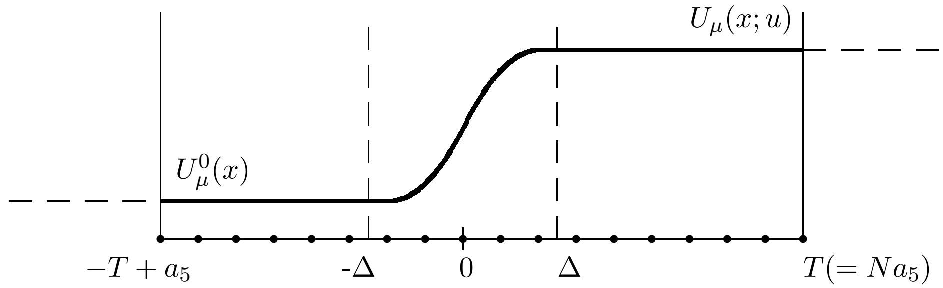

Figure 2: Interpolating five-dimensional gauge field on the lattice

3.3 Inverse five-dimensional Wilson-Dirac operators

For technical reasons, we also require that the five-dimensional Wilson-Dirac operator which couples to the five-dimensional interpolating lattice gauge fields does not have zero mode and is invertible. For this, we asuume the following bound on the $5-\mu$ plaquette,[5]

$$\|1-U_{5\mu}(x,t)\| < \left(\frac{a_5}{a}\right)\epsilon_5, \qquad 30\epsilon + 20\epsilon_5 < \left\{1-|1-m_0|^2\right\}. \tag{3.17}$$

Note that since we can estimate the size of the $5-\mu$ plaquette as

$$\|1-U_{5\mu}(x,t)\| \simeq \frac{a_5}{\Delta}\|U^0_\mu(x)-U^1_\mu(x)\|, \tag{3.18}$$

and

$$\epsilon_5 \simeq \frac{a}{\Delta}\|U^0_\mu(x)-U^1_\mu(x)\|, \tag{3.19}$$

this bound holds true as long as we choose Δ/a large enough. [6] Then the five-dimensional Wilson-Dirac operator is bounded from below by a positive constant,

$$\left\|a^2\left(D_{5\mathrm{w}}-\frac{m_0}{a}\right)^\dagger\left(D_{5\mathrm{w}}-\frac{m_0}{a}\right)\right\| \geq \left\{(1-30\epsilon-20\epsilon_5)^{\frac{1}{2}}-|1-m_0|\right\}^2. \tag{3.20}$$

[5] Y.K. is grateful to D.B. Kaplan for discussions and suggestions on this point.

[6] The bound on the five-dimensional plaquette variables can be regarded as a sufficient condition for the existence of chiral fermions on the boundary walls. In this respect, we note that in the waveguide approach of domain wall fermion [30, 31], Δ/a is set to unity and the bound on the five-dimensional plaquette variables is not satisfied in general.

Given the potitive lower and upper bounds for the five-dimensional Wilson-Dirac operator,

$$\tilde{\alpha} \leq \left\| a^2 \left(D_{5\mathrm{w}} - \frac{m_0}{a} \right)^\dagger \left(D_{5\mathrm{w}} - \frac{m_0}{a} \right) \right\| \leq \tilde{\beta}, \tag{3.21}$$

it follows that the inverse five-dimensional Wilson-Dirac operator decays exponentially at large distance in the fifth dimension [27]:

$$\left\| \left\{ a^2 D_{5\mathrm{w}}^\dagger D_{5\mathrm{w}} \right\}^{-1} (x, s; y, t) \right\| \leq C \exp \left\{ -\frac{\tilde{\theta}}{2} d_5(x, s; y, t) \right\}, \tag{3.22}$$

where

$$C = \frac{4t}{\tilde{\beta} - \tilde{\alpha}} \left(\frac{1}{1-t} \frac{d_5(x, s; y, t)}{2} + \frac{t}{(1-t)^2} \right), \tag{3.23}$$

$$t = e^{-\tilde{\theta}}, \qquad \cosh \tilde{\theta} = \frac{\tilde{\beta} + \tilde{\alpha}}{\tilde{\beta} - \tilde{\alpha}}, \tag{3.24}$$

and $d_5(x, s; y, t) = |x - y|/a + |s - t|/a_5$.

The similar property holds true for the five-dimensional Wilson-Dirac operator which is subject to the Dirichlet boundary condition in the fifth direction. In order to see this, we note the following identity which holds for $s, t \in [-T + a_5, T]$:

$$\begin{aligned} &\frac{1}{D_{5w(T)} - \frac{m_0}{a}} (s, x; t, y) - \frac{1}{D_{5w} - \frac{m_0}{a}} (s, x; t, y) \\ &\qquad = \frac{1}{D_{5w(T)} - \frac{m_0}{a}} V_{(-T+a_5;T)} \frac{1}{D_{5w} - \frac{m_0}{a}} (s, x; t, y), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} V_{(-T+a_5;T)} \quad = \quad & \frac{1}{a_5} \{ -P_L \delta_{s,-T} \delta_{t,-T+a_5} - P_R \delta_{s,-T+a_5} \delta_{t,-T} \\ & -P_L \delta_{s,T} \delta_{t,T+a_5} - P_R \delta_{s,T+a_5} \delta_{t,T} \}. \end{aligned} \tag{3.26}$$

In this identity, the Dirichlet boundary condition at $t = -T + a_5$ and $t = T$ is implemented in the infinite extent of the fifth dimension by adding the

surface interaction term [32]. The derivation of this identity is given in appendix A. By setting $t = T$, we obtain

$$
\begin{aligned}
&\frac{1}{D_{5w(T)} - \frac{m_0}{a}}\,(s,T)\left(1 + P_L\,\frac{1}{D_{5w} - \frac{m_0}{a}}\,(T+a_5,T)\right) \\
&= \frac{1}{D_{5w} - \frac{m_0}{a}}\,(s,T) \\
&\quad - \frac{1}{D_{5w(T)} - \frac{m_0}{a}}(s,-T+a_5)\,P_R\,\frac{1}{D_{5w} - \frac{m_0}{a}}\,(-T,T).
\end{aligned}
\tag{3.27}
$$

We may assume that the correlator

$$
\left(D_{5w} - \frac{m_0}{a}\right)^{-1}(T+a_5,T) \tag{3.28}
$$

has a finite limit when $T \to \infty$. Then we can infer that the inverse of $D_{5w(T)} - \frac{m_0}{a}$ should decay exponentially (up to power corrections in T) at large distance with the same exponent as Eq. (3.22) for $D_{5w} - \frac{m_0}{a}$. Otherwise, it would contradict with Eq. (3.27). Therefore we obtain the following bound in the limit $T \to \infty$ and $|s - T| \to \infty$,

$$
\left\|\frac{1}{D_{5w(T)} - \frac{m_0}{a}}\,(x,s;y,T)\right\| \leq C'\,(T/a_5)^m \exp\left\{-\frac{\tilde{\theta}}{2a_5}|s-T|\right\}, \qquad (|s-T| \to \infty)\,, \tag{3.29}
$$

with a positive constant C' and a positive integer m.

3.4 Definition of the η-invariant on the lattice

In summary, we consider the following lattice implementation of the η-invariant.

$$
\pi\,\overline{\eta} \equiv \lim_{a_5 \to 0}\lim_{T \to \infty} \mathrm{Im}\ \ln\ \det\left(D_{5\mathrm{w(T)}} - \frac{m_0}{a}\right). \tag{3.30}
$$

It utilizes the complex phase of the determinant of the simplified domain-wall fermion which couples to the interpolating five-dimensional gauge field,

$$
S_{\mathrm{DW}} = a_5 \sum_{t=-T+a_5}^{T} a^4 \sum_x \bar{\psi}(x,t)\left(D_{5\mathrm{w}} - \frac{m_0}{a}\right)\psi(x,t), \qquad (0 < m_0 < 2), \tag{3.31}
$$

with the Dirichlet boundary condition in the fifth direction,

$$\psi_R(x,t)\,|_{t=-T}\, = 0, \qquad \psi_L(x,t)\,|_{t=T+a_5}\, = 0. \tag{3.32}$$

$D_{5\mathrm{w(T)}}$ stands for the five-dimensional Wilson-Dirac operator $D_{5\mathrm{w}}$ which is subject to the Dirichlet boundary condition. The five-dimemsional lattice gauge field "interpolates" two four-dimensional lattice gauge field with same topological charge in a finite region $t \in [-\Delta, \Delta]$. It satisfies the bounds on the plaquette variables as

$$\|1 - U_{\mu\nu}(x,t)\| < \epsilon, \tag{3.33}$$

$$\|1 - U_{5\mu}(x,t)\| < \left(\frac{a_5}{a}\right)\epsilon_5, \quad 30\epsilon + 20\epsilon_5 < \left\{1 - |1-m_0|^2\right\}. \tag{3.34}$$

and has a smooth continuum limit in $a_5 \to 0$.

4 The variation of $\overline{\eta}$ with respect to gauge field

Following the analysis in the continuum theory[11], we next examine the variation of $\overline{\eta}$ with respect to the gauge field. For this purpose, we introduce another parameter $u \in [0,1]$ so that it parametrizes the gauge field configuration at $t \geq +\Delta$ from $U^0_\mu(x)$ to $U_\mu(x)$. (Figure 3)

$$U_\mu(x;u), \qquad u \in [0,1]. \tag{4.1}$$

$$U_\mu(x;u=0) = U^0_\mu(x), \quad U_\mu(x;u=1) = U_\mu(x). \tag{4.2}$$

4.1 Summary of result

Before going into technical details, we first summarize our result. The variation of $\overline{\eta}$ with respect to u can be written as the sum of two contributions as follows:

$$\begin{aligned}\frac{d}{du}\,\overline{\eta}\,[U_\mu(x,t;u)] \;&=\; \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\mathrm{ImTr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w}(\infty)}\,\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}} \\ &\quad+\frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}_x\,P_L\,\frac{d}{du}D\,\frac{1}{D}.\end{aligned} \tag{4.3}$$

The first one is the bulk five-dimensional contribution, which depends on the whole interpolating five-dimensional gauge fields. The second one is

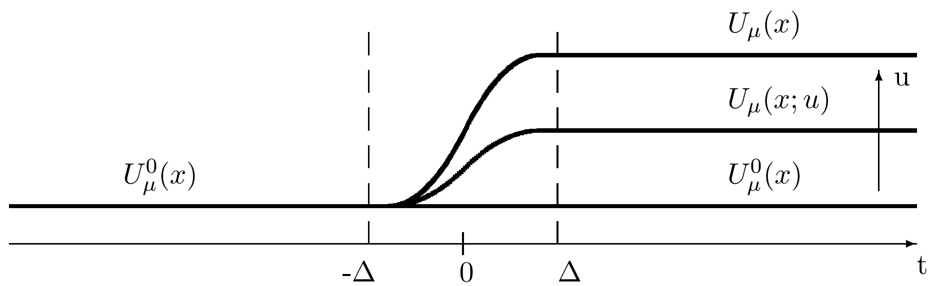

Figure 3: Parameter u

the contribution from the boundaries at $t = -T + a_5$ and $t = T$, which depends only on the boundary values of the interpolating five-dimensional gauge fields.

Remarkably, the surface contribution is expressed by the covariant chiral gauge current associated with Neuberger's Dirac operator. This surface term can be related to the imaginary part of the effective action for the chiral Ginsparg-Wilson fermion which is defined with Neuberger's Dirac operator [8, 10, 17], as we will see later.

The bulk contribution reproduces the Chern-Simons term in the classical continuum limit. First of all, this term is a local functional of $U_\mu(x,t)$. This follows from the property of the inverse five-dimensional Wilson-Dirac operator given by Eq. (3.22), which becomes small exponentially at large distance. Secondly, using a plain wave basis on the lattice, as in the continuum analysis, it can be evaluated as

$$\lim_{a_5\to 0}\lim_{T\to\infty} \mathrm{ImTr}_{(T)}\, \frac{d}{du} D_{5\mathrm{w}(\infty)}\, \frac{1}{D_{5\mathrm{w}(\infty)} - \frac{m_0}{a}}$$
$$\underset{a,a_5\to 0}{=} -\lim_{T'\to\infty} \int d^4x \int_{-T'}^{T'} dt\, \frac{1}{32\pi^2}\, \epsilon_{\mu MNKL} \mathrm{tr} \left\{ \frac{d}{du} A_\mu\, F_{MN} F_{KL} \right\} (x,t;u). \tag{4.4}$$

This quantity can be written as the variation of the Chern-Simons term, up to the local current of Bardeen and Zumino [20] which plays the role to translate the covariant gauge current from the surface contribution to the consistent one [11]. Thus the result in the continuum theory [11] is completely reproduced by our lattice implementation of the η-invariant in

the classical continuum limit.[7]

The coefficient of the Chern-Simons term in this calculation is given by the topological number associated with the free five-dimensional Wilson-Dirac operator,

$$
\begin{aligned}
c &= \frac{1}{5!}\int_{-\pi}^{\pi}\frac{d^5k}{(2\pi)^5}\,\epsilon_{MNIJK}\mathrm{Tr}\left\{\left(\partial_M S^{-1}S\right)\left(\partial_N S^{-1}S\right)\times\right.\\
&\qquad\qquad\left.\left(\partial_I S^{-1}S\right)\left(\partial_J S^{-1}S\right)\left(\partial_K S^{-1}S\right)\right\}(k)\\
&= \frac{i}{8\pi^2},
\end{aligned}
\tag{4.5}
$$

where $S(k)$ is the free propagator of five-dimensional Wilson-Dirac fermion.

$$
S^{-1}(k)=\sum_{M=1}^{5}\left(i\gamma_M\sin k_M+2\sin^2\frac{k_M}{2}\right)-m_0 \quad (0<m_0<2). \tag{4.6}
$$

This result is consistent with the previous calculation of the Chern-Simons current by Golterman, Jansen and Kaplan [33]. The similar quantity in which the fifth momentum is continuous has appeared in the calculation of the axial anomaly [34] of the Ginsparg-Wilson fermion defined with Neuberger's Dirac operator.[8]

4.2 Evaluation of $\frac{d}{du}\overline{\eta}$

4.2.1 Separation of bulk contribution

Now we go into details how to evaluate the variation of $\overline{\eta}$. From the continuum argument of [11], we expect that the variation of $\overline{\eta}$ can be written as the sum of two contributions. The first one is the bulk five-dimensional contribution, which should reproduce a part of the Chern-Simons term. The second one is the contribution from the boundaries at $t=-T+a_5$ and $t=T$, which should be related to the effective action of the chiral fermion. In the context of the domain-wall fermion here, it should be related to the chiral light modes at the boundaries.

By taking the variation of $\overline{\eta}$ with respect to u, we obtain

$$
\frac{d}{du}\overline{\eta} = \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\mathrm{ImTr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w(T)}}\,\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}. \tag{4.7}
$$

[7] We are considering the effective action for the right-handed Weyl fermions.

[8] See [35] for the original calculations of the Chern-Simons term induced from the Wilson-Dirac fermion in three dimensions. For the detail analysis of the chiral Jaccobian of the Ginsparg-Wilson fermion, the authors refer the reader to [36, 37]

Note that since $D_{5(T)}$ is defined in the finite interval of $[-T+a_5, T]$, taking account of the Dirichlet boundary condition, the cyclic property of the trace over the fifth dimension holds true here.

In order to separate the bulk five-dimensional contribution from the boundary contribution, we note the following fact: the bulk term comes from the interval $[-\Delta, \Delta]$ where the interpolating field is varing in t. Then it can also be evaluated from the five-dimensional Dirac fermion (the simplified domain-wall fermion) defined in a slightly larger five dimensional space than $t \in [-T+a_5, T]$, say, $t \in [-T-\Delta T+a_5, T+\Delta T]$. (Figure 4)

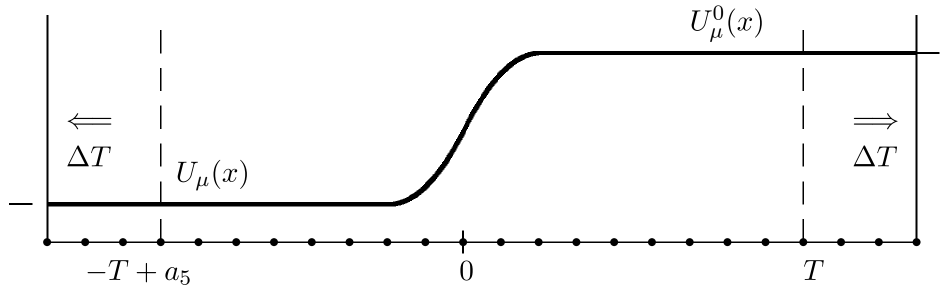

Figure 4: Larger five-dimensional space

The inverse of the Dirac operator in this case

$$\frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}(x, s; y, t) \tag{4.8}$$

does not support the light chiral modes at the original boundaries $t = -T+a_5$ and $t = T$. If we would replace the inverse of the five-dimensional Dirac operator in Eq. (4.7) by Eq. (4.8), then it could include only the bulk contribution. This consideration suggests the following separation:

$$\begin{aligned}
& \frac{1}{\pi}\mathrm{ImTr}_{(T)}\, \frac{d}{du} D_{5\mathrm{w(T)}}\, \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}} \\
= \ & \frac{1}{\pi}\mathrm{ImTr}_{(T)}\, \frac{d}{du} D_{5\mathrm{w(T)}}\, \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}} \\
& + \frac{1}{\pi}\mathrm{ImTr}_{(T)}\, \frac{d}{du} D_{5\mathrm{w(T)}} \left(\frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a} z} - \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}} \right).
\end{aligned} \tag{4.9}$$

Note that we can let ΔT be infinity in this separation.

In order to see that the second term in the r.h.s. of Eq. (4.9) is actually localized at the boundary, we note the following identity which holds for $s,t \in [-T+a_5, T]$:

$$\begin{aligned} & \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}(s,x;t,y) - \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}(s,x;t,y) \\ & \qquad = \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}} V_{(-T+a_5;T)} \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}(s,x;t,y), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} V_{(-T+a_5;T)} &= \frac{1}{a_5}\{-P_L\delta_{s,-T}\delta_{t,-T+a_5} - P_R\delta_{s,-T+a_5}\delta_{t,-T} \\ &\qquad -P_L\delta_{s,T}\delta_{t,T+a_5} - P_R\delta_{s,T+a_5}\delta_{t,T}\}. \end{aligned} \tag{4.11}$$

In this identity,[9] the Dirichlet boundary condition at $t=-T+a_5$ and $t=T$, in the middle of the enlarged extent of the fifth dimension $[-T-\Delta T+a_5, T+\Delta]$, is implemented by adding the surface interaction term [32]. The derivation of this identity is given in appendix A.

Using this identity, the second term can be evaluated as follows:

$$\begin{aligned} & \frac{1}{\pi}\mathrm{ImTr}_{(T)}\frac{d}{du}D_{5\mathrm{w(T)}}\left(\frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}z} - \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}\right) \\ = \ & \frac{1}{\pi}\mathrm{ImTr}_{(T)}\frac{d}{du}D_{5\mathrm{w(T)}}\frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}V_{(-T+1,T)}\frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}} \\ = \ & -\frac{1}{\pi}\mathrm{ImTr}_{(T)}\frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}\right)V_{(-T+a_5,T)}\left(1 - \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}V_{(-T+a_5,T)}\right). \end{aligned} \tag{4.12}$$

Inserting the explicit expression of $V_{(-T+a_5;T)}$, we can evaluate it further to

[9] In the limit $\Delta T \to \infty$, it reduces to Eq. (3.25).

have

$$
\begin{aligned}
= \quad & \frac{1}{\pi}\mathrm{ImTr}_x \, \frac{1}{a_5^2}\left\{ P_L \, \frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(-T+a_5;-T+a_5)} P_R \frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}{}_{(-T;-T)} \right. \\
& \qquad + P_R \, \frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(T;T)} P_L \frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}{}_{(T+a_5;T+a_5)} \\
& \qquad + P_L \, \frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(-T+a_5;T)} P_R \frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}{}_{(T;-T)} \\
& \qquad \left. + P_R \, \frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(T;-T+a_5)} P_L \frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}{}_{(-T;T+a_5)} \right\}.
\end{aligned}
\tag{4.13}
$$

We can see that the first two terms are localized at the boundaries $t = -T + a_5$ and $t = T$, respectively. The last two terms comes from the correlation between two boundaries.

After letting ΔT go to infinity, we can see that the last two terms in Eq. (4.13) vanish in the limit $T \to \infty$, because the inverse of the five-dimensional Wilson-Dirac operator vanishes exponentially for a large separation in the fifth dimension. See Eqs. (3.22) and (3.29).

Therefore, we can write the variation of $\overline{\eta}$ as

$$
\frac{d}{du}\,\overline{\eta} = \frac{d}{du}\,\overline{\eta}_{\mathrm{bulk}} + \frac{d}{du}\,\overline{\eta}_{\mathrm{surf}}, \tag{4.14}
$$

$$
\frac{d}{du}\,\overline{\eta}_{\mathrm{bulk}} = \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\mathrm{ImTr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w}}\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}}, \tag{4.15}
$$

$$
\begin{aligned}
\frac{d}{du}\,\overline{\eta}_{\mathrm{surf}} = \; & \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\mathrm{Im\,Tr}_x\,\frac{1}{a_5^2}\left\{ \right. \\
& P_L\frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(-T+a_5;-T+a_5)} P_R\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}}{}_{(-T;-T)} \\
& \quad \left. + P_R\frac{d}{du}\left(\frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)_{(T;T)} P_L\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}}{}_{(T+a_5;T+a_5)} \right\}.
\end{aligned}
\tag{4.16}
$$

4.2.2 Surface term in the limit $T \to \infty$

We have seen that $\frac{d}{du}\,\overline{\eta}_{\mathrm{surf}}$ is actually localized at the boundaries $t = -T + a_5$ $(-T)$ and $t = T$ $(T + a_5)$. Still it depends on the whole interpolating five-dimensional gauge fields. We next show that in the limit $T \to \infty$, the

interpolating five-dimensional gauge field in the surface contributions can be replaced by the gauge fields of its boundary values.

In order to show this, let us first introduce the five-dimensional gauge fields which are uniform with respect to the fifth-dimensional coordinate t

$$U_\mu^{\leftarrow}(x,t;u) = U_\mu^0(x), \qquad U_\mu^{\rightarrow}(x,t;u) = U_\mu(x;u), \tag{4.17}$$

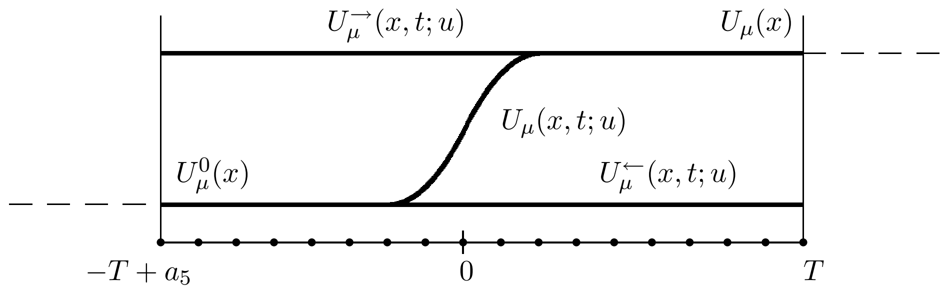

Figure 5: Uniform five-dimensional gauge fields

and consider the five-dimensional Dirac fermions (the simplified domain-wall fermion) which couple to these uniform gauge fields. We denote the five-dimensional Dirac operator of these fermions as $D_5^{\leftarrow}$ and $D_5^{\rightarrow}$, respectively.

In the contribution from the boundary at $t=T$ in Eq. (4.16), the Dirac operator $D_{5\mathrm{w(T)}}$ differs from $D^{\rightarrow}_{5\mathrm{w(T)}}$ only in the region $t\,a_5 \leq +\Delta$. Then we may write as

$$\begin{aligned}
&\frac{1}{D^{\rightarrow}_{5\mathrm{w(T)}} - \frac{m_0}{a}}(T,T) - \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}(T,T) \\
&\quad = \sum_{t' \leq +\Delta} \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}(T,t') \left(D_{5\mathrm{w(T)}} - D^{\leftarrow}_{5\mathrm{w(T)}} \right)(t') \frac{1}{D^{\rightarrow}_{5\mathrm{w(T)}} - \frac{m_0}{a}}(t',T).
\end{aligned} \tag{4.18}$$

Since both $\left(D_{5\mathrm{w(T)}} - \frac{m_0}{a}\right)^{-1}$ and $\left(D^{\rightarrow}_{5\mathrm{w(T)}} - \frac{m_0}{a}\right)^{-1}$ decay exponentially at large distance in the fifth dimension, as shown in Eq. (3.29), we can see that the above difference vanishes exponentially in the limit $T\to\infty$. As for $\left(D_{5\mathrm{w}(\infty)} - \frac{m_0}{a}\right)^{-1}(T+a_5, T+a_5)$, the similar results follow from Eq. (3.22)

and we may replace it by $\left(D^{\rightarrow}_{5(T)}-\frac{m_0}{a}\right)^{-1}(T+a_5,T+a_5)$. By the similar argument, we can show that $D_{5\mathrm{w(T)}}$ and $D_{5\mathrm{w}(\infty)}$ in the contribution from the boundary at $t=-T$ can be replaced by $D^{\leftarrow}_{5\mathrm{w(T)}}$ and $D^{\leftarrow}_{5\mathrm{w}(\infty)}$, respectively.

Furthermore, since the five-dimensional Wilson-Dirac operators depend smoothly on the gauge fields, the differences of $D_{5\mathrm{w(T)}}$ vanishes even after taking the variation with respect to the parameter u. Since $U^{\leftarrow}_{\mu}(x,t;u)$ actually does not depend on u, this implies that the surface term from the boundary at $t=-T+a_5$ vanishes identically.

Thus we have

$$\frac{d}{du}\,\overline{\eta}_{\mathrm{surf}} = \lim_{a_5\rightarrow 0}\lim_{T\rightarrow\infty}\frac{1}{\pi}\mathrm{Im}\,\frac{1}{a_5^2}\,\mathrm{Tr}_x\,P_R\frac{d}{du}\left(\frac{1}{D^{\rightarrow}_{5\mathrm{w(T)}}-\frac{m_0}{a}}\right)(T;T)\,P_L\frac{1}{D^{\rightarrow}_{5\mathrm{w}(\infty)}-\frac{m_0}{a}}(T+a_5;T+a_5). \tag{4.19}$$

4.2.3 Inverse five-dimensional Wilson-Dirac operators in four-dimensional surfaces at boundaries

We next evaluate the inverse five-dimensional Wilson-Dirac operators along four-dimensional surfaces, which appear in the r.h.s. of Eq. (4.19). The inverse of $\left(D^{\rightarrow}_{5(T)}-\frac{m_0}{a_5}\right)$ at $s=t=T$ is nothing but the propagator of the boundary variables of the simplified domain-wall fermion, which we discussed in section 2.3. In the limit $T\rightarrow\infty$, it can be given in terms of the inverse of the effective Dirac operator as follows:

$$\lim_{T\rightarrow\infty} P_R\frac{1}{a_5}\frac{1}{D^{\rightarrow}_{5\mathrm{w(T)}}-\frac{m_0}{a}}(T;T) = P_R\left(\frac{1}{aD_{\mathrm{eff}}}-1\right)[U_\mu(x;u)]\,. \tag{4.20}$$

The inverse of $\left(D^{\rightarrow}_{5(\infty)}-\frac{m_0}{a_5}\right)$ at $s=t=T+a_5$ can be also related to the effective Dirac operator through its representation in terms of the inverse five-dimensional Wilson-Dirac operator, Eq. (2.54). We have

$$\frac{1}{a_5}\frac{1}{D^{\rightarrow}_{5(\infty)}-\frac{m_0}{a}}(T+1;T+1)P_R = \frac{a}{2}\left(\gamma_5 D_{\mathrm{eff}}\gamma_5 - D_{\mathrm{eff}}\right)[U_\mu(x;u)]\,P_R. \tag{4.21}$$

With these results, the surface term can be evaluated further as

$$\begin{aligned}
\frac{d}{du}\overline{\eta}_{\text{surf}} &= \lim_{a_5\to 0}\frac{1}{\pi}\text{Im}\,\text{Tr}_x\, P_R\frac{d}{du}\left(\frac{1}{D_{\text{eff}}}-1\right)P_L\frac{1}{2}\left(\gamma_5 D_{\text{eff}}\gamma_5 - D_{\text{eff}}\right) \\
&= \lim_{a_5\to 0}\frac{1}{\pi}\text{Im}\,\text{Tr}_x\,\frac{d}{du}\frac{1}{D_{\text{eff}}}P_L\,\left(-D_{\text{eff}}\right)\,[U_\mu(x;u)] \\
&= \lim_{a_5\to 0}\frac{1}{\pi}\text{Im}\,\text{Tr}_x\,P_L\,\frac{d}{du}D_{\text{eff}}\,\frac{1}{D_{\text{eff}}}\,[U_\mu(x;u)]\,. \qquad (4.22)
\end{aligned}$$

In the limit $a_5 \to 0$, D_{eff} reduces to the original Neuberger's Dirac operator D of Eq. (2.15) and finally we obtain Eq. (4.3).

4.2.4 Bulk term in the continuum limit

We next turn to the bulk contribution $\frac{d}{du}\overline{\eta}_{\text{bulk}}$ given by Eq. (4.15),

$$\frac{d}{du}\overline{\eta}_{\text{bulk}} = \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\text{ImTr}_{(T)}\,\frac{d}{du}D_{5\text{w}}\frac{1}{D_{5\text{w}(\infty)}-\frac{m_0}{a}}.$$

We will calculate this contribution in the classical continuum limit $a \to 0$ and will show that it reproduces the variation of the Chern-Simons term up to the local current of Bardeen and Zumino [20].

In evaluating the bulk contribution, we set $a_5 = a$. At the same time as to take the continuum limit, we also take the limit of the infinite extent of the fifth dimension $N \to \infty$, keeping $T = Na_5 \gg \Delta$ finite. The limit $T \to \infty$ is taken at last. We adopt the plane wave basis $e^{ik_M(x_M/a)}$, where the five-dimensional coordinate is denoted as $x_M = (x_\mu, t)$ $(M = 1,2,3,4,5)$ with upper case Latin indices.

The five-dimensional Wilson-Dirac operator $D_{5(\infty)}$ acts on the plane wave basis as follows:

$$\begin{aligned}
&\left(D_{5(\infty)} - \frac{m_0}{a}\right)e^{ik_M(x_M/a)} \\
= \;& e^{ik_M(x_M/a)}\left(\sum_{M=1}^{5}\frac{1}{a}\left(i\gamma_M\sin k_M + 2\sin^2\frac{k_M}{2}\right) - \frac{m_0}{a}\right. \\
&\left. - \sum_M \frac{1}{2}\left[(1-\gamma_M)e^{ik_M}\nabla_M + (1+\gamma_M)e^{-ik_M}\nabla_M^\dagger\right]\right). \qquad (4.23)
\end{aligned}$$

The second term in the r.h.s. may be expanded for a smooth background as

$$
\begin{aligned}
V(k) &\equiv \sum_M \frac{1}{2}\left[(1-\gamma_M)e^{ik_M}\nabla_M + (1+\gamma_M)e^{-ik_M}\nabla_M^\dagger\right] \\
&= i\frac{\partial}{\partial k_M} S(k)^{-1}\mathcal{D}_M + \mathcal{O}(a),
\end{aligned} \tag{4.24}
$$

where $\mathcal{D}_M$ is the covariant derivative in the continuum limit

$$
\mathcal{D}_M = \partial_M + iA_M(x) \tag{4.25}
$$

and $S(k)$ is the free propagator of the five-dimensional Wilson-Dirac fermion,

$$
S(k)^{-1} = \sum_{M=1}^{5}\left(i\gamma_M \sin k_M + 2\sin^2\frac{k_M}{2}\right) - m_0 \quad (0 < m_0 < 2). \tag{4.26}
$$

Then, inserting the delta-function

$$
\delta_{xy} = \int_{-\pi}^{\pi} \frac{d^5k}{(2\pi)^5}\, e^{ik_M(x-y)_M/a} \tag{4.27}
$$

into the functional trace of the bulk term, we obtain the expansion

$$
\begin{aligned}
\frac{d}{du}\overline{\eta}_{\text{bulk}} &= \lim_{T'\to\infty}\lim_{a\to 0}\frac{1}{\pi}\mathrm{Im}\mathrm{Tr}_{(T)}\int_{-\pi}^{\pi}\frac{d^5k}{(2\pi)^5}\ \times \\
&\qquad e^{-ik_M(x_M/a)}\frac{d}{du}D_5\frac{1}{D_{5(\infty)}-\frac{m_0}{a}}e^{ik_M(x_M/a)} \\
&= \lim_{T'\to\infty}\lim_{a\to 0}\frac{1}{\pi}\mathrm{Im}\sum_{x_M}\int_{-\pi}^{\pi}\frac{d^5k}{(2\pi)^5}\ \times \\
&\qquad \mathrm{Tr}\,(-1)\frac{d}{du}V(k)\sum_{l=0}^{\infty} a^{l+1}\left\{S(k)V(k)\right\}^l S(k) \\
&= \lim_{T'\to\infty}\frac{1}{\pi}\mathrm{Im}\int\int_{-T'}^{T'} d^4x dt\ C_{JMNKL}\mathrm{Tr}\left\{\frac{d}{du}A_J\,\mathcal{D}_M\mathcal{D}_N\mathcal{D}_K\mathcal{D}_L\right\} \\
&\quad +\mathcal{O}(a),
\end{aligned} \tag{4.28}
$$

where

$$
C_{JMNKL} = \int_{-\pi}^{\pi}\frac{d^5k}{(2\pi)^5}\,\mathrm{tr}(S\partial_J S^{-1})(S\partial_M S^{-1})(S\partial_N S^{-1})(S\partial_K S^{-1})(S\partial_L S^{-1})(k). \tag{4.29}
$$

This coefficient can be evaluated using the fact that it gives a topological number associated with the five-dimensional Wilson-Dirac propagator $S(k)$. [33, 34]. It is completely anti-symmetric and takes the following value:

$$C_{JMNKL} = \epsilon_{JMNKL} \frac{i}{8(\pi)^2}, \tag{4.30}$$

with the convention for the gamma matrices $\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 = 1$. Then the bulk term in the classical continuum limit is given as

$$\frac{d}{du}\,\overline{\eta}_{\text{bulk}} = \lim_{T'\to\infty} -\frac{1}{\pi} \int d^4x \int_{-T'}^{T'} dt\ \frac{1}{32(\pi)^2} \mathrm{Tr}\left\{ \frac{d}{du} A_J\, F_{MN} F_{KL} \right\} + \mathcal{O}(a). \tag{4.31}$$

5 The lattice η-invariant and the effective action for chiral Ginsparg-Wilson fermions

5.1 Relation to the effective action for chiral Ginsparg-Wilson fermions

Now we discuss the relation of our lattice implementation of the η-invariant to the effective action for the chiral Ginsparg-Wilson fermions in abelian and non-abelian chiral gauge theories [8, 10, 17]. For the one-parameter family of the gauge fields of Eq. (4.1), the effective action for the right-handed chiral Ginsparg-Wilson fermion is parametrized by u:

$$\overline{\Gamma}_{\text{eff}} = \ln \det M_{kj} \left[U_\mu(x;u) \right]. \tag{5.1}$$

The variation of the effective action with respect to the parameter u is obtained from Eq. (2.37) by the choice of $\zeta_\mu(x) = \frac{d}{du} U_\mu(x;u)\, U_\mu^{-1}(x;u)$ as

$$\frac{d}{du}\overline{\Gamma}_{\text{eff}} \left[U_\mu(x;u) \right] = \mathrm{Tr} \frac{d}{du} D\, \hat{P}_R D^{-1} P_L + \sum_k \left(v_k, \frac{d}{du} v_k \right). \tag{5.2}$$

Comparing this with the variation of the lattice η-invariant Eq. (4.3), we obtain

$$\begin{aligned} \frac{d}{du}\overline{\Gamma}_{\text{eff}} \left[U_\mu(x;u) \right] &= \frac{d}{du}\,\overline{\eta}\left[U_\mu(x,t;u) \right] \\ &\quad - \lim_{a_5\to 0} \lim_{T\to\infty} \frac{1}{\pi} \mathrm{ImTr}_{(T)}\ \frac{d}{du} D_{5(\infty)}\, \frac{1}{D_{5(\infty)} - \frac{m_0}{a}} \\ &\quad + \sum_k \left(v_k, \frac{d}{du} v_k \right). \end{aligned} \tag{5.3}$$

This equation implies that the combination of the last two terms of the r.h.s. can be writen as a total derivative in u of a certain functional of the five-dimensional gauge field, $U_\mu(x,t;u)$. We denote it by $2\pi\overline{Q}_5\left[U_\mu(x,t;u)\right]$. Then, by integrating in u, we obtain a formula for the imaginary part of the effective action:

$$\mathrm{Im}\overline{\Gamma}_{\text{eff}}\left[U_\mu\right] - \mathrm{Im}\overline{\Gamma}_{\text{eff}}\left[U_\mu^0\right] = \pi\overline{\eta}\left[U_\mu(x,t)\right] + 2\pi\overline{Q}_5\left[U_\mu(x,t)\right], \tag{5.4}$$

where

$$\begin{aligned} 2\pi\overline{Q}_5\left[U_\mu(x,t)\right] \equiv & -\lim_{a_5\to 0}\lim_{T\to\infty}\int_0^1 du\,\mathrm{ImTr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w}(\infty)}\,\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}} \\ & +\int_0^1 du\,\sum_k\left(v_k,\frac{d}{du}v_k\right). \end{aligned} \tag{5.5}$$

This is the relation which may be regarded as the lattice counterpart of Eq. (2.8).

$\overline{Q}_5$ here can be regarded as a lattice expression of the Chern-Simons term in the following sense. 1) First of all, $\overline{Q}_5$ compensates the dependence of $\overline{\eta}$ on the path of the interpolation and make it integrable so that it can give the effective action of chiral fermions, which depends on only the values of gauge fields at the boundaries. 2) $\overline{Q}_5$ reproduces the non-abelian gauge anomaly of the effective action, while $\overline{\eta}$ is gauge invariant. If $U_\mu(x)$ is obtained from $U_\mu^0(x)$ by a certain gauge tranformation,

$$U_\mu(x) = g(x)U_\mu^0(x)g(x+\hat{\mu}a)^{-1}, \tag{5.6}$$

we may consider an interpolation of the gauge transformation function, $g(x,t)$, such that $g(x,t=-\infty)=1$ and $g(x,t=\infty)=g(x)$ and the region of the interpolation is within $t\in[-\Delta,\Delta]$.[10] Then we obtain

$$\begin{aligned} &\mathrm{Im}\,\overline{\Gamma}_{\text{eff}}\left[g(x)U_\mu^0(x)g(x+\hat{\mu}a)^{-1}\right] - \mathrm{Im}\,\overline{\Gamma}_{\text{eff}}\left[U_\mu^0(x)\right] \\ &\qquad = 2\pi\overline{Q}_5\left[g(x,t)U_\mu^0(x)g(x+\hat{\mu}a,t)^{-1}, g(x,t)g(x,t+a_5)^{-1}\right]. \end{aligned} \tag{5.7}$$

We should also note the role of the contribution of the second term of the r.h.s. of Eq. (5.5), so called the measure term. As we have seen,

[10] In this equation, the fifth component of the interpolating five-dimensional gauge field is introduced. The evaluation of $\frac{d}{du}\overline{\eta}$ in section 4 holds true even if we introduce the fifth component of the gauge field as long as its support is within the region $t\in[-\Delta,\Delta]$.

by virtue of the measure term, the path-dependece in the u-integration is removed and $\overline{Q}_5$ becomes a functional of $U_\mu(x,t)$. It corresponds to the local current of Bardeen and Zumino in the continuum theory. More importantly, as shown by Lüscher in [10], the measure term plays a crucial role for the gauge invariance of the effective action in abelian chiral gauge theories on the lattice.

5.2 Gauge invariance of the lattice Chern-Simons term in abelian chiral gauge theories

The gauge-invariant choice of the measure term implies the gauge-invariance of the lattice Chern-Simons term, $\overline{Q}_5$. In order to see this, let us consider the lattice Chern-Simons term, $\overline{Q}_5$, in an abelian gauge theory and examine its gauge transformation property under the infinitesimal gauge transformation,

$$\delta A_\mu(x,t) = -\partial_\mu \omega(x,t), \tag{5.8}$$

$$\omega(x,t=-\infty) = 0, \qquad \omega(x,t=\infty) = \omega(x). \tag{5.9}$$

Since $\overline{\eta}$ is gauge invariant, the transformation of the five-dimensional bulk term of $\overline{Q}_5$ (See Eq. (5.5)) can be evaluated through the transformation of the surface contribution in Eq. (4.3). Then $\overline{Q}_5$ is transformed as follows:

$$\begin{aligned}
&\delta_\omega 2\pi \overline{Q}_5 \left[U_\mu(x,t), U_5(x,t)=1\right] \\
&= -\delta_\omega \int_0^1 du \,\mathrm{Im}\,\mathrm{Tr}_x\, P_L \frac{d}{du} D \,\frac{1}{D} - i \int_0^1 du\, a^4 \sum_x \omega(x) \partial_\mu^* j_\mu(x) \left[uA_\mu\right] \\
&= -i \int_0^1 du \sum_x \omega(x) \left\{ \mathrm{tr}\gamma_5 \left(1-aD\right)(x,x) - a^4 \partial_\mu^* j_\mu(x) \right\} \left[uA_\mu\right]. \qquad (5.10)
\end{aligned}$$

Here we have noted that $\delta_\omega D\left[uA_\mu\right] = iu\left[\omega, D\right]$.

As discussed in section 2.2, for anomaly free abelian chiral theories, the anomalous term which is induced from the five-dimensional bulk term can be written in the form

$$\mathrm{tr}\gamma_5 \left(1-aD\right)(x,x) = a^4 \partial_\mu^* \bar{k}_\mu(x). \tag{5.11}$$

On the other hand, it is possible to choose the measure term so that it satisfies the anomalous conservation law of Eq. (2.41),

$$\partial_\mu^* j_\mu(x) = \partial_\mu^* \bar{k}_\mu(x). \tag{5.12}$$

Then these two terms cancels exactly and the lattice Chern-Simons term $\overline{Q}_5$ becomes gauge-invariant.

As pointed out by Suzuki in [17], the ansatz for the measure term Eq. (2.42) can be obtained from the following integral expression for the effective action:

$$\overline{\Gamma}_{\text{eff}}\left[A_\mu\right] = \int_0^1 dt \left(\operatorname{Tr} P_L \frac{d}{dt} D \frac{1}{D}\left[tA_\mu\right] - ia^4 \sum_x A_\mu(x)\bar{k}_\mu(x)\left[tA_\mu\right] \right). \quad (5.13)$$

Comparing Eq. (5.13) with Eq. (5.2), we find that the integration of the measure term can be given by the integration of the local current $\bar{k}_\mu(x)$:

$$\int_0^1 du \sum_k \left(v_k, \frac{d}{du} v_k \right) = - \int_0^1 du\, a^4 \sum_x A_\mu(x)\bar{k}_\mu(x)\left[uA_\mu\right]. \quad (5.14)$$

Then we obtain a compact expression for the lattice Chern-Simons term as follows:

$$\begin{aligned} 2\pi\overline{Q}_5\left[U_\mu(x,t)\right] \;\equiv\; & -\lim_{a_5\to 0}\lim_{T\to\infty}\int_0^1 du \operatorname{ImTr}_{(T)} \frac{d}{du} D_{5\mathrm{w}(\infty)} \frac{1}{D_{5\mathrm{w}(\infty)} - \frac{m_0}{a}} \\ & - \int_0^1 du\, a^4 \sum_x A_\mu(x)\,\bar{k}_\mu(x)\left[uA_\mu\right]. \end{aligned} \quad (5.15)$$

For the non-abelian gauge theories, the gauge covariant local current such like $\bar{k}_\mu(x)$ in the case of abelian chiral gauge theories is not obtained so far.[11] Such a gauge covariant local current could correct the five-dimensional bulk term

$$\lim_{a_5\to 0}\lim_{T\to\infty}\int_0^1 du \operatorname{ImTr}_{(T)} \frac{d}{du} D_{5\mathrm{w}(\infty)} \frac{1}{D_{5\mathrm{w}(\infty)} - \frac{m_0}{a}} \quad (5.16)$$

to give the lattice Chern-Simons term $2\pi\overline{Q}_5$ with desired properties.

5.3 Integrability of $\overline{\eta}$

In the classical continuum limit, the Chern-Simons term vanishes identically, when the condition for gauge anomaly cancellation is satisfied. Then the effective action for chiral fermions can be given entirely by the η-invariant.

[11] In the recent work [18] by Lüscher, it has been shown to all orders of an expansion in powers of the lattice spacing that the gauge covariant local current of the desired property exists.

Then one may ask whether this ideal situation would happen on the lattice with a finite lattice spacing, when the condition for gauge anomaly cancellation is satisfied,

$$\mathrm{Im}\,\Gamma_{\mathrm{eff}} \stackrel{?}{=} \pi\overline{\eta} \qquad \text{if} \quad \sum_R \mathrm{Tr}\left(T^a\left\{T^b, T^c\right\}\right) = 0. \tag{5.17}$$

For this, the five-dimensional bulk term should vanish identically, or should become integrable and depend only on the boundary values of the interpolating five-dimensional gauge field. It does not seem to be the case in general, however, from the result in the previous subsection. We need further study on this point. It may be possible to realize this ideal case by deforming the five-dimensional Wilson-Dirac operator, which enters to the fermion action Eq. (3.1), as suggested by Neuberger [22, 39].

6 Summary and Discussion

In this paper, we considered a lattice implementation of the η-invariant, using the complex phase of the determinant of the (simplified) domain-wall fermion, which couples to an interpolating five-dimensional gauge field. It is realizing the idea of Kaplan and Schmaltz explicitly on the lattice. The lattice η-invariant is examined and is shown to have a direct relation to the imaginary part of the (gauge invariant) effective action for the chiral Ginsparg-Wilson fermion in the case using Neuberger's Dirac operator. Although the formula of the lattice η-invariant seems to be practical, the issue of the integrability is remained. A lattice expression for the five-dimensional Chern-Simons term is obtained. It should also be examined how the global anomaly [40] fits in this implementation of the η-invariant [41].

Our analysis shows clearly and explicitly that the interplay between the four-dimensional chiral fermion and the five-dimensional (massless) fermion, which is known in the continuum theory, can be realized on the lattice in the framework of the domain-wall fermion and the overlap formalism, where the Ginsparg-Wilson relation is built in. It is expected that other known relations over various dimensions could be also realized in the framework of lattice gauge theory.

Quite recently, starting from the Ginsparg-Wilson relation, a general formula of the effective action for chiral Ginsparg-Wilson fermions is derived by Lüscher [18] and its relation to the η-invariant is suggested. It is conceivable that there is a close relation between this formula and the implementation of the η-invariant discussed in this paper. The relation should be clarified in detail. This issue is under investigation.

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Appendix

A Dirichlet boundary condition by surface interaction

In section 4, in order to show that the surface term $\frac{d}{du}\overline{\eta}$ is localized at the boundary, we use the following identity which holds for $s, t \in [-T+1, T]$:

$$\frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}}(s,x;t,y) - \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}(s,x;t,y) \\ = \frac{1}{D_{5\mathrm{w(T)}} - \frac{m_0}{a}} V_{(-T+a_5;T)} \frac{1}{D_{5\mathrm{w(T+\Delta T)}} - \frac{m_0}{a}}(s,x;t,y) \tag{A.1}$$

where

$$\begin{aligned} V_{(-T+a_5;T)} &= \frac{1}{a_5}\{-P_L\delta_{s,-T}\delta_{t,-T+1} - P_R\delta_{s,-T+1}\delta_{t,-T} \\ &\qquad -P_L\delta_{s,T}\delta_{t,T+1} - P_R\delta_{s,T+1}\delta_{t,T}\}. \end{aligned} \tag{A.2}$$

In this appendix, we give the derivation of this identity.

For this purpose, let us introduce the five-dimensional Dirac fermion defined in the larger five dimensional space $[-T-\Delta T + a_5, T+\Delta T]$, but with the couplings between the lattice sites $(-T, -T+a_5)$ and between the lattice sites $(T, T+a_5)$ omitted.
We denote the five-dimensional Dirac operator of this system by $D^{\vee}_{5(T+\Delta T)}$. Then the difference between $D^{\vee}_{5(T+\Delta T)}$ and $D_{5(T+\Delta T)}$ is given by the surface interaction $V_{(-T+a_5;T)}$ of Eq. (3.26) (Eq. (A.2)).

$$D^{\vee}_{5\mathrm{w(T+\Delta T)}} = D_{5\mathrm{w(T+\Delta T)}} - V_{(-T+a_5;T)}. \tag{A.3}$$

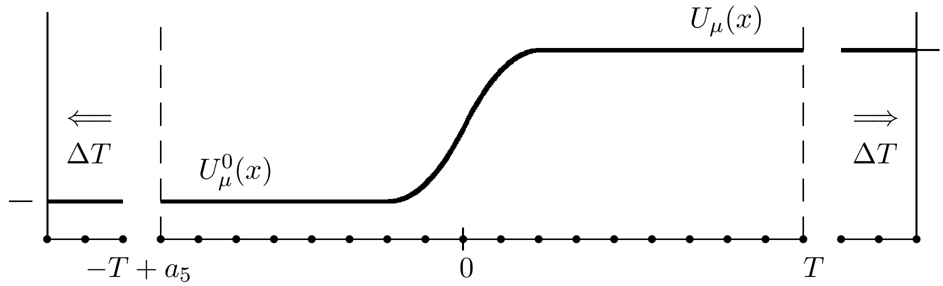

Figure 6: Implementation of Dirichlet B.C. by surface interaction

Then we have

$$\frac{1}{D^{\vee}_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}-\frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}} = \frac{1}{D^{\vee}_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}\,V_{(-T+1;T)}\,\frac{1}{D_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}. \tag{A.4}$$

On the other hand, the field variables in the interval $[-T+a_5, T]$ does not have any coupling to those outside the region and they are nothing but the field variables described by $D_{5\mathrm{w(T)}}$. Then, we have

$$\frac{1}{D^{\vee}_{5\mathrm{w(T+\Delta T)}}-\frac{m_0}{a}}\,(s,x;t,y) = \frac{1}{D_{5\mathrm{w(T)}}-\frac{m_0}{a}}\,(s,x;t,y) \tag{A.5}$$

for $s, t \in [-T+a_5, T]$. From these two relations, Eq. (4.10) (Eq. (A.1)) follows immediately. In the limit $\Delta T \to \infty$, it reduces to Eq. (3.25).

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